

Like a bird on the wire,  
 Like a drunk in a midnight choir  
 I have tried in my way to be free<sup>1</sup>.

*To Stephen Smale, at his 80 - th birthday*

## TURNING WASHINGTON'S HEURISTICS IN FAVOR OF VANDIVER'S CONJECTURE

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ABSTRACT. A famous conjecture bearing the name of Vandiver states that  $h_p^+ = 1$  in the  $p$ -cyclotomic extension of  $\mathbb{Q}$ . Heuristics arguments of Washington, which have been briefly exposed in [La], p. 261 and [Wa], p. 158 suggest that the Vandiver conjecture should be false, if certain conditions of statistical independence are fulfilled. In this note we assume that Greenberg's conjecture is true for the  $p$ -th cyclotomic extensions and prove an elementary consequence of the assumption that Vandiver's conjecture fails for a certain value of  $p$ : the result indicates that there are deep correlations between this fact and the defect  $\lambda^- i(p)$ , where  $i(p)$  is like usual the irregularity index of  $p$ , i.e. the number of Bernoulli numbers  $B_{2k} \equiv 0 \pmod{p}$ ,  $1 < k < (p-1)/2$ . As a consequence, if one combines the various assumptions in Washington's heuristics, these turn, on base of the present result, into an argument in favor of the Vandiver's conjecture.

### 1. INTRODUCTION

Let  $p$  be an odd prime and  $\mathbb{K} = \mathbb{Q}[\zeta]$  be the  $p$ -th cyclotomic field and  $G = \text{Gal}(\mathbb{K}/\mathbb{Q})$ . If  $X$  is a finite abelian group, we denote by  $X_p$  its  $p$ -Sylow group; let  $A = \mathcal{C}(\mathbb{K})_p$ , the  $p$ -Sylow subgroup of the class group  $\mathcal{C}(\mathbb{K})$  and  $h^+, h^-$  the sizes of  $A^+$  respectively  $A^-$ . In a letter to Kronecker from 1857, Kummer refers to  $p \nmid h^+$  as a *noch zu beweisender Satz*, a theorem yet to prove (see also [Wa], p. 158). The fact was stated later as a conjecture by Vandiver.

In [La], p. 261 Washington gives an heuristic argument which suggests that there might be an asymptotic amount of  $O(\log \log(N))$  of primes  $p \leq N$  for which  $\lambda(A_\infty^-) = i(p) + 1$ , where  $i(p)$  is the irregularity index of  $p$ , i.e. the number of Bernoulli numbers  $B_{2k}$ ,  $1 < k < (p-1)/2$

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<sup>1</sup>Leonard Cohen: *Bird on the wire*.

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that vanish modulo  $p$ . In [Wa], p.158, Washington starts with a naive argument, on base of which the cyclotomic unit  $\eta_{2k} := e_{2k}(1 - \zeta)^{\sigma-1}$  (see below for the definition of the idempotents  $e_j \in \mathbb{F}_p[G]$ ) may be a  $p$ -th power with probability  $1/p$ : this yields a probability of more than one half, for the failure of Vandiver's conjecture, so the argument is obviously too crude. Washington considers then that the probabilities that a Bernoulli number vanishes modulo  $p$  and the one that the corresponding cyclotomic unit  $\eta_{2k}$  is a  $p$ -power are independent: this heuristic leads to a frequency of  $O(\log \log(n))$  primes  $p < n$  for which the conjecture fails. As a consequence, various specialists in the field expect that the conjecture should not always hold. Our result in this note, shows that if Vandiver's conjecture fails, then one has the additional condition  $\lambda^- > i(p)$ . If one considers this condition also as statistically independent (!) from the two conditions in Washington's heuristics, then the same argument suggest that there may be  $O(1)$  primes  $p < n$  for which Vandiver's conjecture fails, thus possibly none. *Therefore the elementary result in this note may be understood as one that turns Washington's heuristics into an argument in favor of Vandiver's conjecture.*

In this paper we prove an elementary fact, which implies that the failure of Vandiver's conjecture has an impact on the value of  $\lambda$ , and thus two events which were supposed to be uncorrelated in the heuristic approach: namely  $h_p^+ \neq 0$  and  $\lambda^- > i(p)$  are not independent. No direct consequence can be drawn as to the truth of the Kummer - Vandiver conjecture; however we have an explicite theorem which indicates an unknown dependence, and also a method of investigation which may be extended for the purpose of investigating more possible consequences of the assumption that the Kummer-Vandiver conjecture is false.

The result of this paper is the following:

**Theorem 1.** *Let  $p$  be an odd prime with irregularity index  $i(p) = 1$ . If  $h_p^+ > 1$ , then  $\lambda^- \geq 2$ .*

Since  $\lambda^- > 1$  is an implication of  $h_p^+ > 1$ , the two events cannot be considered as independent events, each one with probability  $1/p$ . But this implication can also suggest that the probability that  $\eta_{2k}$  is a  $p$ -th power has rather the probability  $1/p^2$  than  $1/p$ , since it implies the vanishing of a higher order Bernoulli number. Either way, we consider that our elementary result should suggest that it is worthwhile to consider that Vandiver's conjecture might be true, and pursue the investigation for the reasons why this may be the case. For this purpose,

the central idea of our proof can be extended, with additional detail, to the general case, and this shall be done in a subsequent paper.

Note that we restrict our analysis, for simplicity, to the case of irregularity index 1. However, this is the critical case in Washington's heuristics, and if the assumption of "statistical independence" is close to reality<sup>1</sup>, then the probability of failure of the conjecture for higher values of  $i(p)$  can only be smaller, so the argument stays valid.

## 2. PROOF OF THE THEOREM

We let  $\mathbb{K} = \mathbb{Q}[\zeta]$  be the  $p$ -th cyclotomic extension and  $\mathbb{K}_n = \mathbb{K}[\zeta^{1/p^n}]$ ,  $n \geq 1$ , the  $p^n$ -th extension. The galois groups are

$$\begin{aligned} G = \text{Gal}(\mathbb{K}/\mathbb{Q}) &= \{\sigma_a : a = 1, 2, \dots, p-1, \zeta \mapsto \zeta^a\} \cong (\mathbb{Z}/p \cdot \mathbb{Z})^*, \\ G_n = \text{Gal}(\mathbb{K}_n/\mathbb{Q}) &= G \times \langle \tau \rangle, \quad \tau(\zeta_{p^n}) = \zeta_{p^n}^{1+p}, \end{aligned}$$

so  $\tau$  generates  $\text{Gal}(\mathbb{K}_n/\mathbb{K})$ , in particular. If  $g \in \mathbb{F}_p$  is a generator of  $(\mathbb{Z}/p \cdot \mathbb{Z})^*$ , then  $\sigma = \sigma_g$  generates  $G$  multiplicatively. We write  $j \in G$  for complex multiplication. For  $\sigma \in G$  and  $R \in \{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Z}/(p^m \cdot \mathbb{Z})\}$  we let  $\varpi(\sigma) \in R$  be the value of the Theichmüller character on  $\sigma$ ; for  $R = \mathbb{F}_p$  we may also write  $\hat{\sigma}$  for this values. The orthogonal idempotents  $e_k \in R[G]$  are

$$e_k = \frac{1}{p-1} \sum_{a=1}^{p-1} \varpi^k(\sigma_a) \cdot \sigma_a^{-1}.$$

If  $X$  is a finite abelian  $p$ -group on which  $G$  acts, then  $e_k(\mathbb{Z}_p)$  acts via its approximants to the  $p^m$ -th order; we shall not introduce additional notations for these approximants. A fortiori, complex conjugation acts on  $X$  splitting it in the canonical plus and minus parts:  $X = X^+ \oplus X^-$ , with  $X^+ = X^{1+j}$ ,  $X^- = X^{1-j}$ . The units of  $\mathbb{K}$  and  $\mathbb{K}_n$  are denoted by  $E, E_n$  and the cyclotomic units by  $C, C_n$ . The Iwasawa invariants  $\lambda, \lambda^-$  are related to the cyclotomic  $\mathbb{Z}_p$ -extension  $\mathbb{K}_\infty = \bigcup_n \mathbb{K}_n$  and  $A_n = (C(\mathbb{K}_n))_p$  are the  $p$ -parts of the ideal class groups of  $\mathbb{K}_n$ . They form a projective sequence with respect to the relative norms  $N_{m,n} = \mathbf{N}_{\mathbb{K}_m/\mathbb{K}_n}$ ,  $m > n \geq 1$  and  $\mathbf{A} = \varprojlim_n A_n$ . We also write  $A$  for  $A_1$ . We shall write for simplicity  $A(\mathbb{L}) = (\mathcal{C}(\mathbb{L}))_p$  for the  $p$ -part of the class group of an arbitrary number field  $\mathbb{L}$ , so  $A = A(\mathbb{K})$ , etc.

We fix now an odd prime  $p$  such that

1. Greenberg's conjecture holds for  $p$ , so  $A^+$  is finite and  $\lambda^+ = 0$ .
2. Vandiver's conjecture fails for  $p$ .

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<sup>1</sup>Washington mentions explicitly that this is the crucial and critical in the various heuristics of this kind.

3. There is a unique irregular index  $2k$  such that  $A_{p-2k} = \varepsilon_{p-2k}A \neq \{1\}$ . Additionally  $A_{2k} \neq \{1\}$ , as a consequence of 2.

Under these premises, we show that  $\mathbb{Z}_p\text{-rk}(\varepsilon_{p-2k}\mathbf{A}) > 1$ , which is the statement of the theorem. We prove the statement by contraposition, so we assume that  $\mathbb{Z}_p\text{-rk}(\varepsilon_{p-2k}\mathbf{A}) = 1$ . Since there is a unique irregular index, the minimal polynomial of  $\mathbf{A}$  is linear. Let  $\mathbb{H}_n/\mathbb{K}_n$  be the maximal  $p$ -abelian unramified extensions. They split in plus and minus parts according to  $A_n = A_n^+ \oplus A_n^-$  and our assumption implies that  $\mathbb{H}_n^+/\mathbb{K}_n$  are cyclic extensions of degree

$$d_n := [\mathbb{H}_n^+ : \mathbb{K}_n] = |A_n^+|.$$

We may also consider  $\mathbb{H}_n^+$  as the compositum of  $\mathbb{K}_n$  with the full  $p$ -part of the Hilbert class field of  $\mathbb{K}_n^+ \subset \mathbb{K}_n$ , the maximal real subextension of  $\mathbb{K}_n$ : thus  $\mathbb{H}_n^+$  is a canonical subfield, corresponding by the Artin map to  $A_n^+$ . It follows that  $\mathbb{H}_n^+/\mathbb{K}_n^+$  is an abelian extension, and thus  $\mathbb{H}_n^+$  is a CM field (see also [Wa], Lemma 9.2 for a detailed proof). There is a canonic construction of radicals from  $A_n^-$ , such that  $\mathbb{H}_n \cdot \mathbb{K}_m \subset \mathbb{K}_m[(A_m^-)^{1/p^m}]$  for sufficiently large  $m$ . As a consequence of Greenberg's conjecture holding for  $\mathbb{K}$ , there is an  $n_0 \geq 1$  such that  $|A_n^+| = |A_{n_0}^+|$  for all  $n \geq n_0$  and for such  $n$ , let  $a_n \in A_n^-$  generated this cyclic group. Let  $\Omega \in a_n$  and  $\alpha_0 \in \mathbb{K}_n^\times$  with  $(\alpha_0) = \Omega^{\text{ord}(a_n)}$ ; there is an  $\alpha = \eta \cdot \alpha_0^{1-j}$ ,  $\eta \in \mu_{p^n}$  which is well defined up to roots of unity, such that  $\mathbb{H}_n^+ \subset \mathbb{K}_n[\alpha^{1/p^n}]$ . The radical  $B_n$  of  $\mathbb{H}_n^+$  is then the multiplicative group generated by  $\alpha$  and  $(\mathbb{K}_n^\times)^{d_n}$ .

Since  $\mathbb{H}_n^+/K_n$  is cyclic, a folklore result, which we prove for completeness in Lemma 3 of the Appendix below, implies that

$$(1) \quad A(\mathbb{H}_n^+) = (\mathcal{C}(\mathbb{H}_n^+))_p = \{1\}.$$

A classical result, proved by Iwasawa [Iw] in a general cohomological language, states that for an arbitrary galois extension  $\mathbb{L}/\mathbb{F}$  of finite number fields, there is a canonical isomorphism

$$(2) \quad H^1(\text{Gal}(\mathbb{L}/\mathbb{F}), E(\mathbb{L})) \cong \mathcal{A}(\mathbb{L}),$$

where  $\mathcal{A}(\mathbb{F})$  are Hilbert's *ambig* ideals, i.e. the ideals of  $\mathbb{L}$  which are invariant under  $\text{Gal}(\mathbb{L}/\mathbb{F})$ , factored by the principal ideals of  $\mathbb{F}$ . These can be either totally ramified ideals or ideals from  $\mathbb{F}$  that capitulate completely (become principal) in  $\mathbb{L}$ . We shall in the sequel often consider the homology groups  $H^0, H^1$  for the unit groups. We can then write, for simplicity

$$H^i(\mathbb{L}/\mathbb{F}) := H^i(\text{Gal}(\mathbb{L}/\mathbb{F}), E(\mathbb{L})) \quad \text{for } i = 0, 1.$$

The isomorphism above restricts also to one of  $p$ -parts of the respective groups; furthermore, complex conjugation also induces canonical isomorphisms of the plus and minus parts of  $H^i$ . The extensions  $\mathbb{H}_n^+/\mathbb{K}_n$  being cyclic of degree  $d_n$ , the Herbrand quotient is  $d_n$  and thus

$$H^1(\mathbb{H}_n^+/\mathbb{K}_n) = d_n \cdot H^0(\mathbb{H}_n^+/\mathbb{K}_n).$$

We claim that  $(H^0(\mathbb{H}_n^+/\mathbb{K}_n))^+ = \{1\}$ . Indeed, the ambig ideals in an unramified extension are capitulated ideals. In our case, since  $d_n = |A_n^+|$  by definition, we have exactly  $|\mathcal{A}(\mathbb{H}_n^+)| = d_n^2$ . This follows from the fact that the plus part capitulates completely, while the minus part is cyclic too and generates the radical of the extension. Consequently  $\Omega^{\text{ord}}(a_n)/d_n$  becomes principal in  $\mathbb{H}_n^+$ , which confirms that

$$|\mathcal{A}(\mathbb{H}_n^+)| = d_n^2 \quad \text{and} \quad |\mathcal{A}(\mathbb{H}_n^+)|^- = |\mathcal{A}(\mathbb{H}_n^+)|^+ = d_n.$$

Therefore,  $|H^0(\mathbb{H}_n^+/\mathbb{K}_n)| = d_n$ . The roots of unity  $\zeta_{p^n} \notin \mathbf{N}_{\mathbb{H}_n^+/\mathbb{K}_n}(E(\mathbb{H}_n^+))$ : indeed, if  $\zeta_{p^m} = N(\delta)$  for  $\delta \in E(\mathbb{H}_n^+)$  and  $m \leq n$ , then  $\varepsilon = \delta/\bar{\delta}$  is well defined in the CM field  $\mathbb{H}_n^+$  and it is a root of unity, by Dedekind's unit Theorem – so  $\varepsilon \in \mathbb{K}_n$ . Moreover, we have  $\mathbf{N}_{\mathbb{H}_n^+/\mathbb{K}_n}(\varepsilon) = \varepsilon^{d_n} = \zeta_{p^m}^2$ . Since  $p$  is odd, it follows that  $\mu_{p^n}/\mu_{p^n/d_n} \subset H^0(\mathbb{H}_n^+/\mathbb{K}_n)$ ; by comparing orders of the groups, we conclude that

$$H^0(\mathbb{H}_n^+/\mathbb{K}_n) = \mu_{p^n}/\mu_{p^n/d_n} = (H^0(\mathbb{H}_n^+/\mathbb{K}_n))^-.$$

We have proved:

**Lemma 1.** *Notations being like above,*

$$(H^0(\mathbb{H}_n^+/\mathbb{K}_n))^+ = \{1\}.$$

*In particular*

$$(3) \quad \mathbf{N}_{\mathbb{H}_n^+/\mathbb{K}_n}(E^+(\mathbb{H}_n^+)) = E^+(\mathbb{K}_n),$$

*where for a CM field  $\mathbb{F}$  we write  $E^+(\mathbb{F}) = \{e \cdot \bar{e} : e \in E(\mathbb{F})\}$ .*

In our case,  $E^+$  are the real units and the units of  $\mathbb{K}_n^+$ , resp.  $\mathbb{H}(\mathbb{K}_n^+) \subset \mathbb{H}_n^+$ ; the prime  $p$  is odd and we are interested in  $p$ -parts, so the implicit exponent 2 in the above definition has no further consequences: the norm is surjective on the real units in our class field. Then  $\mathbb{H}_{n+1}^+ = \mathbb{K}_{n+1} \cdot \mathbb{H}_n^+$  and we have a commutative diagram of fields. By computing  $H^0(\mathbb{H}_{n+1}^+/\mathbb{K}_n)$  in two ways, over the intermediate field  $\mathbb{K}_{n+1}$  and respectively over  $\mathbb{H}_n^+$ , we obtain from (3) that

$$(4) \quad (H^0(\mathbb{K}_{n+1}/\mathbb{K}_n))^+ = (H^0(\mathbb{H}_{n+1}^+/\mathbb{H}_n^+))^+.$$

The core observation of this proof is

**Proposition 1.** *Let  $\lambda_n = 1 - \zeta_{p^n}$ . Then the ramified prime  $\wp_n = (\lambda_n) \subset \mathbb{K}_n$  above  $p$  splits totally in  $\mathbb{H}_n^+$  in  $p$ -principal ideals. Moreover, if  $\nu \in \text{Gal}(\mathbb{H}_n^+/\mathbb{K}_n)$  is a generator of this cyclic group, then there is a prime  $\pi_n \in \mathbb{H}_n^+$  with  $\mathbf{N}_{\mathbb{H}_n^+/\mathbb{K}_n}(\pi_n) = \lambda_n$ .*

*Proof.* Since  $\wp_n = (\lambda_n)$  is principal, the Principal Ideal Theorem implies that it splits completely in the unramified extension  $\mathbb{H}_n^+/\mathbb{K}_n$  and since  $A(\mathbb{H}_n^+) = \{1\}$  by (3), the primes above  $\wp_n$  are  $p$ -principal. Let  $E'(\mathbb{F})$  denote the  $p$ -units of the number field  $\mathbb{F}$ , i.e. the units of the smallest ring containing  $E(\mathbb{K})$  and in which all the primes above  $p$  are invertible. In particular  $E'_n = E_n[1/\lambda_n]$ ; it is customary to denote by  $A'_n$  the  $p$ -part of the ideal class group of the  $p$ -integers. Since  $\mathbb{H}_n^+/\mathbb{K}_n$  splits the prime above  $p$  and  $A_n^+ = (A'_n)^+$  and  $H^0(\text{Gal}(\mathbb{H}_n^+/\mathbb{K}_n), E'(\mathbb{H}_n^+)) = H^0(\mathbb{H}_n^+/\mathbb{K}_n)$ . In particular, the norm  $\mathbf{N}_{\mathbb{H}_n^+/\mathbb{K}_n} : E'(\mathbb{H}_n^+) \rightarrow E'(\mathbb{K}_n)$  is surjective, so there is a prime  $\pi_n \in \mathbb{H}_n^+$  mapping on  $\lambda_n$ .

The proof of this proposition is made particularly simple by the use of (3). However, a more involved proof shows that the facts hold in more generality and the primes above  $\lambda$  are principal in any subfield of the Hilbert class field  $\mathbb{H}_n$ .  $\square$

As a consequence of the proposition, we see that  $\mathcal{A}(\mathbb{H}_n^+) = \{1\}$ . Indeed, by Lemma 3, the class group  $A(\mathbb{H}_n^+) = \{1\}$ , and the only primes that ramify in  $\mathbb{H}_{n+1}^+/\mathbb{H}_n^+$  lay above  $p$ , so they are principal by Lemma 1. There are consequently no real ambig ideals in  $\mathbb{H}_n^+$ . On the other hand, for  $n$  such that  $|A_n^+| = |A_{n+1}^+|$ , the capitulation kernel  $P_n := \text{Ker}(\iota_{n,n+1} : A_n^+ \rightarrow A_{n+1}^+)$  is an  $\mathbb{F}_p$ -space of dimension  $d = p - \text{rank}(\mathbf{A}^+) = p - \text{rank}(A_n) = 1$ . We obtain a contradiction with (4), which shows that if  $i(p) = 1$ , then either  $\mathbb{Z}_p\text{-rk}(A^-) > 1$  or  $h_p^+ = 1$ . This completes the proof of the Theorem.

### 3. THREE DETAILED VARIANTS OF THE PROOF

Let  $m$  be the first index for which  $|A_m^+(\mathbb{K})| = |A_{m+1}^+(\mathbb{K})| = |A^+(\mathbb{K})|$ . Consider now  $\mathbb{H}_n^+ = \mathbb{H}_m \cdot \mathbb{K}_n^+$  for  $n \geq m$ . Since  $[\mathbb{H}_n^+ : \mathbb{K}_n^+] = q = |A(\mathbb{K})^+|$  for all  $n > m$ , we see that  $\text{Ker}(A^+(\mathbb{H}_m) \rightarrow A^+(\mathbb{H}_\infty)) = 1$ .

Iwasawa's Theorem 12 in [Iw1] implies that  $H^1(\text{Gal}(\mathbb{H}_n/\mathbb{H}_m), E'(\mathbb{H}_n)) = 1$ , and the fact that the primes above  $p$  are principal leads by a computation of Herbrand quotients to  $E(\mathbb{H}_m) = N_{n,m}(E(\mathbb{H}_n))$  for all  $n > m$ . We have

$$\begin{aligned} N_{\mathbb{H}_n, \mathbb{K}_m^+}(E(\mathbb{H}_n)) &= N_{\mathbb{H}_m/\mathbb{K}_m^+}(E(\mathbb{H}_m)) = E(\mathbb{K}_m^+) \\ &= N_{n,m}(N_{\mathbb{H}_n/\mathbb{K}_n^+}(E(\mathbb{H}_n))) = N_{n,m}(E(\mathbb{K}_n^+)). \end{aligned}$$

It follows thus that  $N_{n,m}(E(\mathbb{K}_n^+)) = E(\mathbb{K}_m^+)$  for all  $n > m$ . However, for sufficiently large  $n$  we have  $N_{n,m}(E(\mathbb{K}_n^+)) = C(\mathbb{K}_m^+)$  and this would require that  $E(\mathbb{K}_m^+) = C(\mathbb{K}_m^+)$ , which is the contradiction above.

We give below<sup>2</sup> some additional detail about the units in  $\mathbb{H}_n$  which lead to two variants of this proof and reveal interesting details.

**3.1. Metacyclotomic units.** Let  $E(\mathbb{H}_n)$  be the units of  $\mathbb{H}_n^+$  and  $E(\mathbb{K}_n), C(\mathbb{K}_n)$  the units, resp. cyclotomic units of  $\mathbb{K}_n$ . We shall construct some norm coherent sequences of units in  $E(\mathbb{H}_n)$  and relate them to the cyclotomic. For this, fix some large integer  $N$ . We assume for simplicity that the primes above  $p$  in  $\mathbb{H}_n$  are principal and derive some specific units on base of this assumption. Since we are only interested in  $p$ -parts, raising to a power coprime to  $p$  can be neglected.

We shall denote the norms  $\mathbf{N}_{\mathbb{H}_n/\mathbb{H}_m} = \mathbf{N}_{\mathbb{K}_n/\mathbb{K}_m}$  by  $N_{n,m}$  and let  $\mathcal{N} = \mathbf{N}_{\mathbb{H}_n/\mathbb{K}_n}$  for all  $n$  sufficiently large. We choose  $\pi_{2N} \in \mathbb{H}_{2N}$  a prime with  $\mathcal{N}(\pi_{2N}) = 1 - \zeta_{p^{2N}}$  and let  $\pi_n = N_{2N,n}(\pi_{2N})$  for all  $n < 2N$ . The inertia group  $I(\pi_{2N}) \cong \text{Gal}(\mathbb{K}_{2N}/\mathbb{Q})$ . We let  $\sigma', \tau' \in I(\pi_{2N})$  be lifts of  $\sigma, \tau \in \text{Gal}(\mathbb{K}_{2N}/\mathbb{Q})$ , the generators of the subgroups with  $p-1$  and  $p^{2N-1}$  elements, respectively. We identify  $T = \tau' - 1$  a lift of  $T$  to  $\text{Gal}(\mathbb{H}_{2N}/\mathbb{H}_m)$ , where  $m$  is an integer such that  $|A^+(\mathbb{K}_m)| = |A^+(\mathbb{K}_{m+1})| = q$ , say.

We fix an other large integer  $M \gg N$  and define the lifted idempotents

$$\varepsilon'_j = \frac{1}{p-1} \sum_{\varsigma \in \langle \sigma' \rangle} \omega^j(\varsigma) \varsigma^{-1} \in \mathbb{Z}[\text{Gal}(\mathbb{H}_n/\mathbb{Q})],$$

where  $\omega : \langle \sigma' \rangle \rightarrow \mathbb{Z}$  approximates the Teichmüller character to the power  $p^M$ . We also let

$$\tilde{\varepsilon}_j = \frac{1}{p-1} \sum_{\varsigma \in \langle \sigma' \rangle} \varpi^j(\varsigma) \varsigma^{-1} \in \mathbb{Z}_p[\text{Gal}(\mathbb{H}_n/\mathbb{Q})],$$

with  $\varpi$  the  $p$ -adic Teichmüller character.

Then we define  $E^{(2j)}(\mathbb{K}_n^+) = \varepsilon_{2j} E(\mathbb{K}_n^+)$ , so  $\tilde{\varepsilon}_{2j}(E(\mathbb{K}_n^+)/E^{p^M}(\mathbb{K}_n^+)) = E^{(2j)}(\mathbb{K}_n^+)/ (E^{(2j)}(\mathbb{K}_n^+))^{p^M}$ . Moreover,  $m$  verifies for all  $n > m$

$$(5) \quad E(\mathbb{K}_n) = C(\mathbb{K}_n) \cdot E(\mathbb{K}_m), \quad \forall n > m.$$

Indeed, this condition is equivalent to  $[E(\mathbb{K}_n) : C(\mathbb{K}_n)] = |A_n^+| = [E(\mathbb{K}_m) : C(\mathbb{K}_m)] = |A_m^+|$  for all  $n > m$ , a condition which is implied by

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<sup>2</sup>I thank Hiroki Takahashi for some critical questions and an interesting dialogue, which suggested that the details provided in this section may be useful for readers interested to understand better the specific conditions of the  $p$ -th cyclotomic extensions, compared for instance with some real quadratic extensions in which Greenberg's conjecture is known to hold, yet  $A^+ = p$  and  $A^-$  is cyclic.

our assumption (e.g. Fukuda). It follows also that  $[E(\mathbb{K}_l) : C(\mathbb{K}_l)] = |A_l^+| = p|A_{l-1}^+|$  for  $1 < l \leq m$ , and if  $\varepsilon_l \in e_{2k}E(\mathbb{K}_l)$  generates this component up to  $p^M$ -th powers, then

$$E(\mathbb{K}_n) = \langle \varepsilon_m, \eta_n \rangle_{\mathbb{Z}[T]}.$$

We show

**Lemma 2.** *Notations being like above, there is a module  $\overline{C}(\mathbb{H}_n) \subset E(\mathbb{H}_n)$ , of finite index and norm coherent for all  $n \leq 2N$ . Moreover, if  $m$  is the smallest index such that  $|A_m^+| = |A_{m+1}^+|$ , then*

$$(6) \quad E(\mathbb{H}_n) = E(\mathbb{H}_m) \cdot \overline{C}(\mathbb{H}_n) \quad \forall n \geq m.$$

*Proof.* Let  $\eta_n = (\pi_n \cdot \overline{\pi_n})^{\sigma'-1}$  with  $\pi_n$  defined above, let  $\delta_n = (\pi_n \cdot \overline{\pi_n})^T$  and  $C(\mathbb{H}_n) = \eta_n^{\mathbb{Z}[\text{Gal}(\mathbb{H}_n/\mathbb{Q})]}$  be the  $\mathbb{Z}$ -module generated by  $\eta_n$ , for all<sup>3</sup>  $n < N$ . By definition,  $\mathcal{N}(C(\mathbb{H}_n)) = C(\mathbb{K}_n)$  and the modules  $C(\mathbb{H}_n)$  are norm coherent. In particular, if  $e \in C(\mathbb{H}_l)$  then there is a unit  $e' \in C(\mathbb{H}_{2N})$  with  $e = N_{2N,l}(e')$ .

Let further  $(a_n)_{n \in \mathbb{N}} \in A_n^+ \setminus (A_n^+)^p$  be a norm coherent sequence of classes; thus  $a_n^q = 1$  for all  $n \geq m$  and  $q = \exp(A^+)$ . Let  $\mathfrak{Q} \in a_{2N}$  be a totally split prime and  $\gamma \in \mathbb{H}_{2N}$  be such that  $\mathfrak{Q}\mathcal{O}(\mathbb{H}_{2N}) = (\gamma)$ ; if  $\mathfrak{Q}_n = N_{2N,n}(\mathfrak{Q})$  and  $\gamma_n = N_{2N,n}(\gamma)$ , then  $\mathfrak{Q}_n\mathcal{O}(\mathbb{H}_n) = (\gamma_n)$  for all  $n < 2N$ . We let  $c_n = \gamma_n^s \in E(\mathbb{H}_n)$ , a norm coherent sequence of units for  $n \leq 2N$ . Let  $\mathcal{C}(\mathbb{H}_n) = c_n^{\mathbb{Z}[s]} \cdot \delta_n^{\mathbb{Z}[T,s]}$ . Then  $\mathcal{C}(\mathbb{H}_n)$  also form a norm coherent sequence and we define  $\overline{C}(\mathbb{H}_n) = C(\mathbb{H}_n) \cdot \mathcal{C}(\mathbb{H}_n)$ , which is a further norm coherent sequence of units in  $E(\mathbb{H}_n)$ .

We note that  $\text{Ker}(\mathcal{N} : E(\mathbb{H}_n) \rightarrow E(\mathbb{K}_n^+)) = E^s(\mathbb{H}_n) \cdot \mathcal{C}(\mathbb{H}_n)$ . Letting  $\mathcal{E}(\mathbb{H}_n) = E(\mathbb{H}_n)/(E^s(\mathbb{H}_n) \cdot \mathcal{C}(\mathbb{H}_n))$ , it follows from (3) that

$$\mathcal{E}(\mathbb{H}_n) \cong E(\mathbb{K}_n),$$

and the norm  $\mathcal{N}$  is an isomorphism between these two modules. As a consequence, if  $c_0 \in \mathcal{C}(\mathbb{H}_n)$  generates this module and  $e_{2j} \in E(\mathbb{H}_n)$ ,  $j = 0, 1, \dots, \frac{p-3}{2}$  have non trivial image in  $\mathcal{E}$  and map to a set of units  $d_j$ ;  $j = 0, 1, \dots, (p-3)/2$  which generate  $E(K_n^+)$  as a  $\mathbb{Z}[T]$ -module and such that  $d_j$  generates  $\varepsilon_{2j}E(\mathbb{K}_n^+)/E(\mathbb{K}_n^+)^{p^M}$  for fixed, arbitrarily large  $M > 0$ , then

$$\langle c_0; e_0, \dots, e_{(p-3)/2} \rangle_{\mathbb{Z}[T,s]} = E(\mathbb{H}_n).$$

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<sup>3</sup>We define the *metacyclotomic* units only for  $n < N$ , in order to make sure that they are norms from a large extension above  $\mathbb{H}_n$ . By avoiding the limit process  $\cap_N N_{N,n}(E'(\mathbb{H}_n))$  we obtain in addition explicit uniformizers  $\pi_n$ , which are norm coherent.



We let  $e_n \in E(\mathbb{H}_n)$  be such that  $\mathcal{N}(e_n)$  generates  $\varepsilon_{2j}E(\mathbb{K}_n)/E^{p^M}(\mathbb{K}_n)$  for some fixed, large  $M$ . We then consider the systems

$$\begin{aligned}\mathcal{H}_n &= \left\{c_n^{s^j} : j = 0, 1, \dots, p-2\right\} \cup \left\{\delta_n^{s^j T^l} : j = 0, 1, \dots, p-1; l = 0, \dots, p^{n-1}-1\right\} \\ &\quad \cup \left\{\varepsilon'_{2j} \eta_n^{s^i \cdot T^l} : i = 0, 1, 2, \dots, p-1; j = 1, 2, \dots, \frac{p-3}{2}; l = 0, \dots, p^{n-1}-1\right\}, \\ \overline{\mathcal{H}}_n &= \left\{c_n^{s^j} : j = 0, 1, \dots, p-2\right\} \cup \left\{\delta_n^{s^j T^l} : j = 0, 1, \dots, p-1; l = 0, \dots, p^{n-1}-1\right\} \\ &\quad \cup \left\{(e_n^{(2j)})^{s^i \cdot T^l} : i = 0, 1, 2, \dots, p-1; j = 1, 2, \dots, \frac{p-3}{2}; l = 0, \dots, p^{n-1}-1\right\}.\end{aligned}$$

One verifies that  $|\mathcal{H}_n| = \mathbb{Z}\text{-rk}(E(\mathbb{H}_n))$ ; an application of the Nakayama lemma to  $\overline{\mathcal{C}}(\mathbb{H}_n)/\overline{\mathcal{C}}(\mathbb{H}_n)^{p^M}$  implies that the system is also a generating system for  $\overline{\mathcal{C}}(\mathbb{H}_n)$ . Moreover  $\overline{\mathcal{H}}_n$  generates  $E(\mathbb{H}_n)/E^{p^M}(\mathbb{H}_n)$  for  $n \leq m$ . These systems are reminiscent of Hilbert's relative units in his Theorem 91. It follows that  $\overline{\mathcal{C}}(\mathbb{H}_n)$  has finite index in  $E(\mathbb{H}_n)$  for all  $n < 2N$ .

Assume now that (6) is false, and there is some  $m' > m$  and  $e_{m'} \in E(\mathbb{H}_{m'}) \setminus E(\mathbb{H}_m) \cdot \overline{\mathcal{C}}(\mathbb{H}_{m'})$ ; we can assume without restriction of generality that  $e_{m'} \in E^{(2j)}(\mathbb{H}_{m'})$  (see below for the definition of the *components*  $E^{(2j)}(\mathbb{H}_{m'})$ ), since at least one component verifies the condition. Then, by (5)

$$\mathcal{N}(e_{m'}) \in E(\mathbb{K}_m) \cdot \overline{\mathcal{C}}(\mathbb{K}_{m'}) = \mathcal{N}(E(\mathbb{H}_m) \cdot \overline{\mathcal{C}}(\mathbb{H}_{m'})).$$

There is thus a unit  $e \in E(\mathbb{H}_m) \cdot \overline{\mathcal{C}}(\mathbb{H}_{m'})$  such that  $e_{m'} = e \cdot w$ ,  $w \in \text{Ker}(\mathcal{N} : E(\mathbb{H}_{m'}) \rightarrow E(\mathbb{K}_{m'}))$ . Since  $\mathcal{C}_n$  are norm coherent and  $(1 - \varepsilon'_0) \text{Ker}(\mathcal{N} : E(\mathbb{H}_{m'}) \rightarrow E(\mathbb{K}_{m'})) = (1 - \varepsilon'_0)E(\mathbb{H}_{m'})^s$ , we can assume that  $w \in (1 - \varepsilon'_0)E(\mathbb{H}_{m'})$ , so  $w = (e'_{m'})^s$  with  $e'_{m'} \in (1 - \varepsilon'_0)E(\mathbb{H}_{m'})$ . By repeating the same argument, we find that  $e'_{m'} = e'' \cdot w'$ , where  $w' \in \text{Ker}(\mathcal{N} : E(\mathbb{H}_{m'}) \rightarrow E(\mathbb{K}_{m'}))$ . It follows by induction that  $e_{m'} \in E(\mathbb{H}_m) \cdot \overline{\mathcal{C}}(\mathbb{H}_{m'}) \cdot E(\mathbb{H}_{m'})^{s^u}$  for any  $u > 0$ . Since  $s^p = pv$ ,  $v \in (\mathbb{Z}_p[s])^\times$ , it follows that  $e_{m'} \in E(\mathbb{H}_m) \cdot \overline{\mathcal{C}}(\mathbb{H}_{m'}) \cdot E(\mathbb{H}_{m'})^{p^{M'}}$  for arbitrary  $M' > 0$ . But the index  $[E(\mathbb{H}_{m'}) : \overline{\mathcal{C}}(\mathbb{H}_{m'})]$  is finite, so the claim must be true. Therefore  $m$  is defined as the smallest index after which  $|A_n^+|$  stabilizes  $\square$

We obtain the following explicite variant of the first proof given above. Since  $E(\mathbb{H}_m) \neq \overline{\mathcal{C}}(\mathbb{H}_m)$  while the quotient of these modules is finite, there is a unit  $e \in E(\mathbb{H}_m) \setminus \overline{\mathcal{C}}(\mathbb{H}_m)$ , such that  $e^p \in \overline{\mathcal{C}}(\mathbb{H}_m)$ . Consequently, there is a metacyclotomic unit  $c_{m+1} \in \overline{\mathcal{C}}(\mathbb{H}_{m+1})$  such that  $N_{m+1,m}(c_{m+1}) = e^p$ . Let  $d = c_{m+1}/e \notin \overline{\mathcal{C}}(\mathbb{H}_{m+1})$ , by choice of  $e$ . By definition,  $N_{m+1,m}(d) = 1$  and thus Hilbert 90 implies that  $d = \gamma^{\omega_m}$  for some  $\gamma \in \mathbb{H}_{m+1}$ . The ideal  $(\gamma)$  is ambig, and since  $A^+(\mathbb{H}_{m+1}) = 1$ ,

it follows that  $\gamma$  must be a product of principal ramified primes. Thus  $\gamma = \pi_{m+1}^x \cdot \delta$ , for some  $x \in \mathbb{Z}[\text{Gal}(\mathbb{H}_{m+1}/\mathbb{Q})]$  and  $\delta \in E(\mathbb{H}_{m+1})$ . But then  $\pi_{m+1}^{x\omega_{m+1}} \in \overline{C}(\mathbb{H}_{m+1})$  by definition of  $\overline{C}$ . Moreover, the structure identity (6) implies that  $\delta^{\omega_m} \in \overline{C}(\mathbb{H}_{m+1})^{\omega_m} \subset \overline{C}(\mathbb{H}_{m+1})$ . Altogether, we find that  $d \in \overline{C}(\mathbb{H}_{m+1})$  and thus  $e \in \overline{C}(\mathbb{H}_{m+1})$  too. This contradicts the choice  $e \notin \overline{C}(\mathbb{H}_m)$ , given the (6): units which are metacyclotomic in  $\mathbb{H}_{m+1}$  must be so already in  $\mathbb{H}_m$ . Note that this contradiction makes explicite the argument of Iwasawa used in the first proof.

**3.2. Local units and components.** We consider the structure of the local units  $U(\mathbb{K}_n), U(\mathbb{H}_n)$ . For the first, it is known that  $U(\mathbb{K}_n^+) = \bigoplus_k \varepsilon_{2k} x_n^\Lambda$  are  $\Lambda$ -cyclic modules, generated by some  $x_n \in U(\mathbb{K}_n^+)$  (e.g. [Wa], §8). Since  $p$  is totally split, we also have  $U(\mathbb{H}_n) \cong (U(\mathbb{K}_n^+))^p$ , as cartesian product. Let  $\xi_n = (x_n, 1, 1, \dots, 1)$  under the Chinese Remainder Theorem, with the first component being the projection to  $\mathbb{H}_{n, \wp_n}$  and  $\wp_n$  a ramified prime above the initially chosen  $\wp = (\pi)$ . Then  $\tilde{\varepsilon}_{2j} \xi_n$  defines the  $2j$ -component of  $U(\mathbb{H}_n)$  as a  $\Lambda[s]$ -module. Since  $\nu$  acts transitively on the projections in  $U(\mathbb{H}_n)$ , it follows that these modules are also *canonically* given by

$$(7) \quad U_{2j}(\mathbb{H}_n) = (U(\mathbb{K}_n^+))^p$$

where the right hand side is a cartesian of copies corresponding to the completions at the various primes above  $p$ . The action of  $\text{Gal}(\mathbb{K}_n^+/\mathbb{Q})$  by conjugation on  $\nu$  implies

$$\nu^\sigma = \nu^{\varpi(\sigma)^{2k}}, \quad \nu^\tau = \nu^{p+1},$$

with  $\varpi$  being the Teichmüller character. From this, one verifies without using the construction above, that  $(U(\mathbb{K}_n^+))^p$  is a canonic  $\mathbb{Z}_p[\text{Gal}(\mathbb{H}/\mathbb{Q})]$ -module which well defined by (7). We say that  $U_{2j}(\mathbb{H}_n)$  is the  $2j$ -th component of  $U(\mathbb{H}_n)$  and have

$$U(\mathbb{H}_n) = \bigoplus_{j=0}^{(p-3)/2} U_{2j}(\mathbb{H}_n).$$

Let  $\mathbf{N}_l = \cap_n N_{n,l}(U(\mathbb{H}_n))$  for  $l \geq 1$ . Then class field theory implies that  $U(\mathbb{H}_l)/\mathbf{N}_l \cong \prod U^{(1)}(\mathbb{Z}_p)$  is the product of the  $p$  copies of  $U^{(1)}(\mathbb{Z}_p)$  in  $U(\mathbb{H}_l)$ . Since  $\mathbb{Z}_p \subset U_0(\mathbb{K}_n^+)$  it follows from the definition (7) that  $U(\mathbb{H}_l)/\mathbf{N}_l \hookrightarrow U_0(\mathbb{H}_n)$ , and therefore

$$(8) \quad U_{2j}(\mathbb{H}_m) \subset \mathbf{N}_m \quad \text{for all } j \neq 0.$$

We identify the units  $E(\mathbb{H}_n)$  with their *diagonal embedding*  $E(\mathbb{H}_n) \hookrightarrow U(\mathbb{H}_n)$ . Since  $E(\mathbb{H}_n)/E(\mathbb{H}_n)^{p^M}$  is a  $\mathbb{Z}_p$ -module, we can use the components of the local units for defining the *components* of global units

by

$$\begin{aligned} E^{(2j)}(\mathbb{H}_n) &= E(\mathbb{H}_n) \cap \left( U_{2j}(\mathbb{H}_n) \cdot E(\mathbb{H}_n)^{p^M} \right), \\ C^{(2j)}(\mathbb{H}_n) &= E^{(2j)}(\mathbb{H}_n) \cap \overline{C}(\mathbb{H}_n) \end{aligned}$$

By definition of  $U_{2j}$ , we have  $E^{(2l)}(\mathbb{H}_n) \cap E^{(2j)}(\mathbb{H}_n) = E(\mathbb{H}_n)^{p^M}$  for  $l \neq j$ .

Let now  $e \in E^{(2k)}(\mathbb{H}_m) \setminus N_{m+1,m}(E^{(2k)}(\mathbb{H}_{m+1}))$ . Note that by (6), the existence of such a unit is granted. But then, by applying  $\mathcal{N}$  we deduce that  $\varepsilon_{2k}E(\mathbb{K}_m) = \varepsilon_{2k}C(\mathbb{K}_m)$ . However, our initial assumption implies that  $\varepsilon_{2k}A_m^+ \neq \{1\}$  and the main conjecture thus leads to  $[\varepsilon_{2k}E(\mathbb{K}_m) : \varepsilon_{2k}C(\mathbb{K}_m)] \neq 1$ . Therefore, we conclude that there is unit  $e$  with the claimed properties. The Hasse norm principle implies by the above that  $e \in N_{m+1,m}(\mathbb{H}_{m+1}^\times)$ . Let thus  $x \in \mathbb{H}_{m+1}^\times$  with  $N_{m+1,m}(x) = e$ . Then  $(x) = \mathfrak{X}^{\omega_m}$ , since  $H^1(\langle \nu \rangle, I(\mathbb{H}_n))$  vanishes for the ideals in the cyclic extension  $\mathbb{H}_n/\mathbb{K}_n$ . Moreover,  $\mathfrak{X}$  is not  $p$ -principal: otherwise, if  $(t, p) = 1$  and  $\mathfrak{X}^t = (\xi)$  is principal, then  $x^t = \xi^{\omega_m} \cdot e_1$  for some unit  $e_1 \in E(\mathbb{H}_{m+1})$ . If the  $ut + vp = 1$  we have

$$e = N_{m+1,m}(e^v \cdot \xi^{\omega_m} e_1) = N_{m+1,m}(e^v \cdot e_1) \in N_{m+1,m}E(\mathbb{H}_{m+1}),$$

and we had assumed  $e \notin N_{m+1,m}E(\mathbb{H}_{m+1})$ . Consequently  $A(\mathbb{H}_{m+1}) \neq \{1\}$ , which is a contradiction to the Lemma 3 that completes the proof.

Note that universal norms behave differently in the positive components and in the zero component, the latter being by (8) the one which localizes the norm residue defects along the cyclotomic  $\mathbb{Z}_p$ -extension of any base field. This fact may indicate the major distinction between zero components and positive indexed components.

#### 4. APPENDIX

For the sake of completeness, we give a proof of the following

**Lemma 3.** *Let  $\mathbb{K}$  be a number field and  $A$  be the  $p$ -part of its class group, while  $\mathbb{H}$  is the  $p$ -part of its Hilbert class group. If  $A$  is cyclic, then  $A(\mathbb{H}) := \mathcal{C}(\mathbb{H})_p = \{1\}$ .*

*Proof.* Since  $A$  is cyclic,  $\text{Gal}(\mathbb{H}/\mathbb{K}) \cong A$  is a cyclic group and the ideals of a generating class  $a \in A$  are inert and become principal in  $\mathbb{H}$ . Let  $\sigma = \varphi(a) \in \text{Gal}(\mathbb{H}/\mathbb{K})$  be a generator and  $s = \sigma - 1$ . Suppose that  $b \in A(\mathbb{H}) \setminus A(\mathbb{H})^{(s,p)}$  is a non trivial class and let  $\mathfrak{Q} \in b$  be an ideal above a rational prime  $\mathfrak{q} \subset \mathbb{K}$ , which splits completely in  $\mathbb{H}/\mathbb{K}$ : such a prime must exist, by Tchebotarew's Theorem. If  $b' = [\mathfrak{q}]$ , then the order of  $b'$  must be a power of  $p$ , since this holds for  $\mathfrak{Q}$ ; thus  $b' \in A(\mathbb{K}) = \langle a \rangle$ . But we have seen that the ideals from  $a$  capitulate in

$\mathbb{H}$ , so  $b = 1$ , in contradiction with our choice. Therefore  $b' = 1$  and thus  $\mathbf{N}_{\mathbb{H}/\mathbb{K}}(b) = 1$ . Furtwängler's Hilbert 90 Theorem for ideal class groups in cyclic extensions [Fu] says that  $\text{Ker}(\mathbf{N} : A(\mathbb{H}) \rightarrow A(\mathbb{K})) \subset A(\mathbb{H})^s$ , and this implies that  $b \in A(\mathbb{H})^s$ , which contradicts the choice of  $b$  and completes the proof.  $\square$

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